A Study on Invariants
in Fuzzy Convexity Theory

F.W.Levi introduced abstract convexity theory in 1951. It was further developed by many authors like D.C.Kay, E.W.Womble, R.E.Jamison-Waldner, G.Sierksma and M.van de Vel. Notion of fuzziness in concrete convexity was introduced by M.D.Weiss. It was later studied by A.K.Katsaras, D.B.Liu, R.Lowen and others.

Various notions of dependence of a nonempty finite set in a convexity space were studied by Levi, Soltan, M.van de Vel and others. This leads to the introduction of convex invariants like Helly number, Caratheodory number, Radon number and ex-change number in a convexity space X. Levi gave the first general definition of the invariants Helly number and Radon number. In 1975, Sierksma introduced the exchange number as a modifica-tion of the Caratheodory number. Kay and Womble developed a general theory of convex invariants in 1971. M.van de Vel investi-gated the behavior of these invariants under the formation of CP images, CC images and subspaces.

In 1968, J. Eckhoff introduced the concept of convex product space. The invariants in a convex product space are studied by

Gerard Sierksma. In 1981, he introduced the notion of a convex sum space and determined the convex invariants in the sum space when the universal sets X_1, X_2, ..., X_n are disjoint.

Fuzziness in abstract convexity was introduced and studied by M.V.Rosa in 1994. She defined a fuzzy convexity space as a pair (X, C) with X, a set and C, a family of fuzzy subsets of X called fuzzy convex sets satisfying the conditions:

1. $\underline{0}, \underline{1} \in C$

2. If $F \subseteq C$, then $\wedge F \in C$

3. If $F \subseteq C$ is non-empty and totally ordered by inclusion, then $\vee F \in C$.

She also defined fuzzy convex to convex (FCC) maps and fuzzy convexity preserving (FCP) maps in fuzzy convexity spaces in the following way. Let (X, C_1) , (Y, C_2) be fuzzy convexity spaces. Let $f : X \to Y$. Then f is said to be a

1. fuzzy convexity preserving function (FCP) if for each fuzzy convex set K in Y , f^{-1} (K) is a fuzzy convex set in X.
2. fuzzy convex to convex function (FCC) if for each fuzzy convex set K in X , f (K) is a fuzzy convex set in Y.

Shiny Philip, in 2010, introduced the notion of fuzziness to

different types of dependence of a nonempty finite set in a con-vexity space.

Let (X, C) be a fuzzy convexity space. A nonzero finite fuzzy subset F of X is

1. Helly dependent (or H-dependent) if $\bigwedge_{a_\alpha \in F} Co(F \backslash a_\alpha) \neq \underline{0}$

   where $F \backslash a_\alpha = F \wedge a^0_\alpha$. Otherwise, it is H-independent.
2. Radon dependent (or R-dependent) if there exists a Radon

   partition $\{F_1, F_2\}$ of F (i.e., $F_1 \wedge F_2 = \underline{0}$; $F_1 \vee F_2 = F$) such that $Co(F_1) \wedge Co(F_2) \neq \underline{0}$.

   Otherwise, it is R-independent.
3. Caratheodory dependent (or C-dependent) provided

   $Co(F) \leq \bigvee_{a_\alpha \in F} Co(F \backslash a_\alpha)$. Otherwise, it is C-independent.
4. exchange dependent (or E-dependent) if for each $p_\alpha \in F$, $Co(F \backslash p_\alpha) \leq \bigvee [Co(F \backslash a_\beta)$; $a_\beta \in F \backslash p_\alpha; a \neq p]$. Otherwise, it is E-independent.

She also defined the fuzzy convex invariants namely Helly num-ber, Caratheodory number, Radon number and exchange number in a fuzzy convexity space as follows: Let (X, C) be a fuzzy con-vexity space.

1. The Helly number of X, denoted by h(X), is the smallest ‘ n ’ such that each

nonzero finite fuzzy subset F of X, with cardinality of its support at least ‘ n+1 ’, is Helly dependent.

2. The Caratheodory number of X, denoted by c(X), is the smallest ‘ n’ such that each nonzero finite fuzzy subset F of X, with cardinality of its support at least ‘ n + 1’, is Caratheodory dependent.

3. The Radon number of X is the smallest ‘ n’ such that each nonzero finite fuzzy subset F of X, with cardinality of its support at least ‘ n + 1’, is Radon dependent. It is denoted by r(X) or by r.

4. The exchange number of X is the smallest ‘ n’ such that each nonzero finite fuzzy subset F of X, with cardinality of its support at least ‘ n + 1’, is exchange dependent. It is denoted by e(X) or by e.

The purpose of this thesis is to study the relationship between these fuzzy convex invariants in fuzzy convexity spaces. The in-vestigation is extended to the product and disjoint sum of fuzzy convexity spaces. In chapter 1, a brief introduction to fuzzy con-vexity theory is given. Some basic definitions and properties of fuzzy convexity spaces are also included.

In chapter 2, some relations between fuzzy convex invariants are established. It includes the characterization of Caratheodory number in fuzzy convexity spaces. Levi inequality is proved in the fuzzy context. It states that if (X, C) is a fuzzy convexity space with Helly number h and Radon number r then $h \leq r$.

Sierksma inequality is also proved in the fuzzy context which states that $e - 1 \leq c \leq \max\{h, e - 1\}$ where e, c and h are the exchange number, Caratheodory number and Helly number in a fuzzy convexity space.

The behavior of fuzzy convex invariants under the formation of FCP images, FCC images and subspaces are also discussed in this chapter and the following conclusions are arrived at:

1. If (X, C_1) , (Y, C_2) are two fuzzy convexity spaces then for a surjective FCP function f : $X \to Y$, $h(X) \geq h(Y)$ and $r(X) \geq r(Y)$.
2. If f is also FCC, then $c(X) \geq c(Y)$ and $e(X) \geq e(Y)$.
3. If Y is a fuzzy subspace of the fuzzy convexity space (X, C) then $c(X) \geq c(Y)$ and $e(X) \geq e(Y)$.
4. If Y is also convex in X, then $h(X) \geq h(Y)$; $r(X) \geq r(Y)$.

Chapter 3 is a detailed study of fuzzy convex invariants in a fuzzy convex product space. Firstly, it is proved that the Helly number of a fuzzy convex product space is the supremum of the Helly numbers of the factor spaces. The next result gives a com-plete description of the Caratheodory number of a product space in terms of the Caratheodory number and exchange number of the factor spaces:

Let (X_1, C_1), (X_2, C_2) be two fuzzy convexity spaces and $X = X_1 \times X_2$ be the fuzzy convex product space. If c_i and e_i are the Caratheodory number and exchange number of X_i for i = 1, 2 then the Caratheodory number c of the product space is determined as follows:

1. If $c_i < e_i$ for $i = 1, 2$ then $c = c_1 + c_2$
2. If $c_i < e_i$ for exactly one of $i = 1, 2$ then $c = c_1 + c_2 - 1$
3. If $c_i \geq e_i$ for $i = 1, 2$ then $c = c_1 + c_2 - 2$

In case of the exchange number e of the product space, the following conclusions are arrived at:

1. If $c_i < e_i$ for $i = 1,2$ then $e \geq e_1 + e_2 - 1$
2. If $c_1 < e_1$ and $c_2 \geq e_2$ then $e \geq e_1 + c_2 - 1$

3. If $c_1 \geq e_1$ and $c_2 < e_2$ then $e \geq c_1 + e_2 - 1$

4. If $c_i \geq e_i$ for $i = 1,2$ then $e \geq c_1 + c_2 - 1$.

In chapter 4, the notion of a disjoint fuzzy convex sum space, which is analogous to the definition of G. Sierksma in (crisp) convexity theory, is introduced. In this chapter, the various fuzzy convex invariants in a disjoint fuzzy convex sum space are derived as follows.

Let (X_1, C_1) , (X_2, C_2) be two fuzzy convexity spaces and $X = X_1 + X_2$, the disjoint fuzzy convex sum space. If h_i, c_i, e_i, r_i are the Helly number, Caratheodory number, exchange number, Radon number of (X_i, C_i) for i = 1, 2 and h, c, e and r are those of the sum space X then

1. $h = h_1 + h_2$ 2. $c = \max \{c_1, c_2\}$

3. $e = c + 1$ 4. $r = r_1 + r_2$.

Shiny Philip introduced interval spaces in the fuzzy context and studied some properties. In chapter 5, certain concepts like denseness, straightness and Ramification property of fuzzy inter-val spaces are discussed. It is proved that a straight, fuzzy inter-val space with decomposable fuzzy intervals has the Ramification property. But fuzzy interval spaces having the Ramification property need not be either decomposable or straight. It is illus-trated with an example.

Fuzzy join operators are defined and some properties stud-ied in this chapter. Moreover, a fuzzy Bryant-Webster space is defined as a dense, straight, JHC fuzzy convexity space with de-composable segments and proved that a fuzzy join space is a fuzzy Bryant-Webster space.

Note:

Some of the results have appeared in the following journals:

1. International Mathematical Forum, 6(20) (2011), 995–1004.
2. Applied Mathematical Sciences, 5(35) (2011), 1719–1725.
3. International Mathematical Forum, 6(57) (2011), 2815–2822.
4. International Mathematical Forum, 7(20) (2012), 999–1005.

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